Norwegian School of Economics and Business Administration

ON STRUCTURAL MODELS OF DEBT

Hayne Leland
University of California, Berkeley

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I want to focus on three specific topics

1) ***What are the shortcomings of "traditional" structural models in explaining credit spreads and default probabilities?***

2) ***Can jump processes and/or liquidity premiums remedy these problems?***

3) ***What are the resulting implications for corporate decisions, with a specific focus on optimal leverage choice.***

Let's start with a typical "traditional" model…

(this has many predecessors, starting with Merton (1974))

Key elements:

1) *Asset value process* under the *risk-neutral measure*

$$\boxed{dV(t)/V(t) = (r-\delta)\,dt + \sigma\,dW(t)}$$

…a diffusion process with continuous sample path, where

$V(t)$	asset value (value of cash flows) at time t
r	risk-free interest rate, assumed constant through time
δ	fractional (of value) payout rate to all securities, a constant
σ	asset volatility, also a constant
$dW(t)$	increment to a Wiener process at time t

$V(0) = V_0$

2) *Debt*

Characterized by ***principal P***, ***coupon flow C***, ***maturity T***

Other important parameters are ***default cost fraction*** α and ***tax rate*** τ (implying the after-tax coupon cost is $(1 - \tau)C$)

Exponential debt model: for debt issued at time $t = 0$

> Debt principal is *retired at a proportional rate* $m = 1/T$ (e.g. through sinking fund)

> This implies that debt principal and coupon are *exponentially declining*; thus remaining principal, coupon of debt issued at $t = 0$ are $e^{-mt}P$, $e^{-mt}C$

> This also implies that the *average maturity of debt* $= 1/m = T$.

> Retired debt is replaced by *newly-issued debt with same principal, coupon, and maturity*; thus total P, C, T remain *constant through time*.

> Total debt service flow is constant $C + mP$, unless default

RISK NEUTRAL VALUATION OF DEBT

- The ***discounted expected value of current debt's cash flow*** under the risk neutral measure is

$$\boxed{D = \int_0^\infty e^{-rt}[e^{-mt}(C+mP)](1-F)dt + (1-\alpha)V_B \int_0^\infty e^{-rt}e^{-mt} f\, dt} \qquad (1)$$

where F is the *cumulative distribution function of first passage time* from V_0 to a default barrier V_B, and f is its *density function.*

Integrating the first term of (1) by parts gives

$$\boxed{D = \frac{C+mP}{r+m}(1-\int_0^\infty e^{-(r+m)t} f\, dt) + (1-\alpha)V_B \int_0^\infty e^{-(r+m)t} f\, dt} \qquad (2)$$

- We now use the ***only mathematical result*** we will need for the paper.

For processes with constant drift g and volatility σ:

The expected present value of \$1 received at first passage to default V_B (from value V_0 at t = 0), when discounted at an arbitrary rate z, is

$$\int_0^{\infty} e^{-zt} f(t;V_0,V_B)dt = \left(\frac{V_0}{V_B}\right)^{-y(g,z)},$$

where

$$y(g,z) = \frac{(g-.5\sigma^2)+((g-.5\sigma^2)^2+2z\sigma^2)^{0.5}}{\sigma^2} \qquad (3)$$

Using (3), the value of debt in equation (2) is

$$\boxed{D = \frac{C+mP}{r+m}\left(1-\left(\frac{V_0}{V_B}\right)^{-y_1}\right) + (1-\alpha)V_B\left(\frac{V_0}{V_B}\right)^{-y_1}} \qquad (4)$$

where $y_1 = y(g, z)$ in (3) when $g = r - \delta$ and $z = r + m$.

NB: when $m = 0$ (infinite life debt), (4) is the same formula as in Leland (1994).

- ***We can also readily compute closed form solutions for***
  - > The value of equity E
  - > The total value of firm leveraged firm $v = D + E$.
- ***The endogenous optimal default boundary V_B, satisfies*** the smooth-pasting conditions $\frac{\partial E(V;V_B)}{\partial V}\Big|_{V=V_B} = 0$

- ***The optimal endogenous default barrier V_B is:***

$$\boxed{V_B = \frac{\dfrac{(C+mP)y_1}{(r+m)} - \dfrac{\tau C y}{r}}{1+(1-\alpha)y_1+\alpha\, y}} \qquad (5)$$

where $y = y(g, z)$ in (3) when $g = r - \delta$ and $z = r$.

- ***Substituting for V_B into* (4) *gives closed form solution for D (and E and v).***

Default probabilities can be easily calculated:

Cumulative first passage times to V_B, with $g = r - \delta + \pi$

where π = *asset risk premium* ➔ g = *actual asset growth rate*

HOW WELL DOES THE MODEL PREDICT? CALIBRATION:

TABLE 2: CALIBRATION OF MODEL PARAMETERS

	Rating			Sources
	A	**Baa**	**B**	
Leverage *D/v*	**32.0%**	**43.3%**	**65.7%**	HH; CGH
Average Debt Maturity *T*	**10 yrs.**	**7.5 yrs.**	**5 yrs.**	HH; Duffee, Stohs & Maurer
Asset Volatility σ	**22%**	**22%**	**31%**	Schaefer & Strebulaev (2004)
Payout Rate δ	**6%**	**6%**	**6%**	HH (avg. of dividends, coupons 1973-98)
Tax Advantage to Debt τ	**15%**	**15%**	**15%**	Leland & Toft (1996), Graham (2003)
Default Costs α	**30%**	**30%**	**30%**	Consistent with recovery rates, all ratings
Asset Risk Premium	**4%**	**4%**	**4%**	Consistent with asset beta about 0.6, all ratings
Recovery Ratio	**60%**	**50%**	**40%**	EG (60.6%, 49.4%, 37.5%); HH (51.3% for all)

EG = Elton & Gruber (2001), HH = Huang & Huang (2003), CGH = Collin-Dufresne, Goldstein & Helwege (2003)

Using these parameters, let’s see how well model matches ***observed spreads*** from H&H, E&G, and Duffee over 1985-1995, and ***default data*** from Moody’s over the period 1970-2000.

>> Unlike H&H, we do ***not*** *choose volatilities to match default rates*

HOW WELL DOES THE MODEL DO?   NOT WELL!!

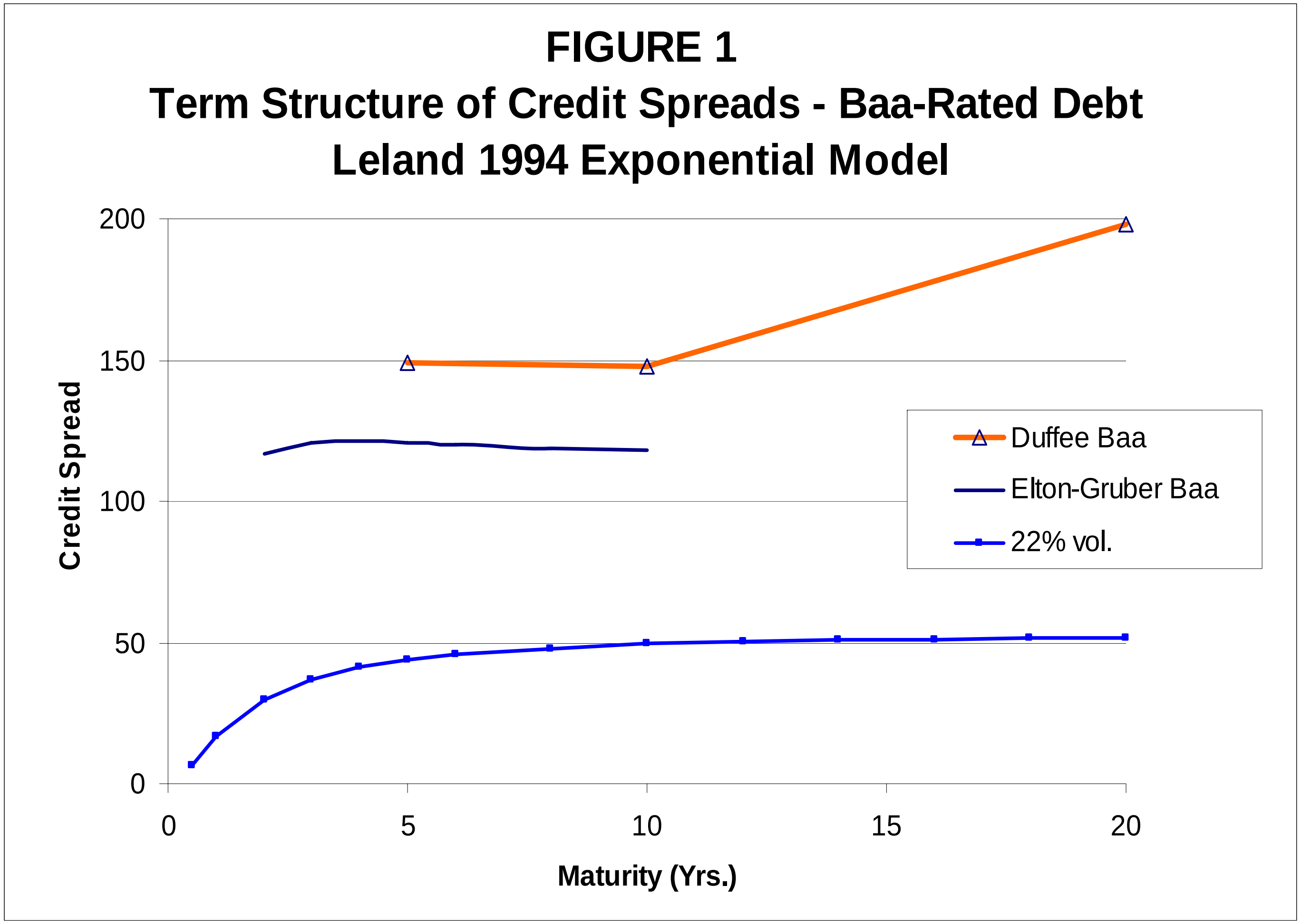


FIGURE 1 shows model predicts Baa spreads that are about 1/3 of actual. . .

- Confirms most empirical studies that traditional structural models ***underestimate spreads.*** (e.g. Jones, Mason, Rosenfeld (1984), Huang & Huang (2004))

- But a widely-cited article by Eom, Helwege and Huang (*EHH*, 2004) claims that the structural model of Leland and Toft (*LT*, 1996)

  ––***substantially overestimates*** spreads, even at short maturities.

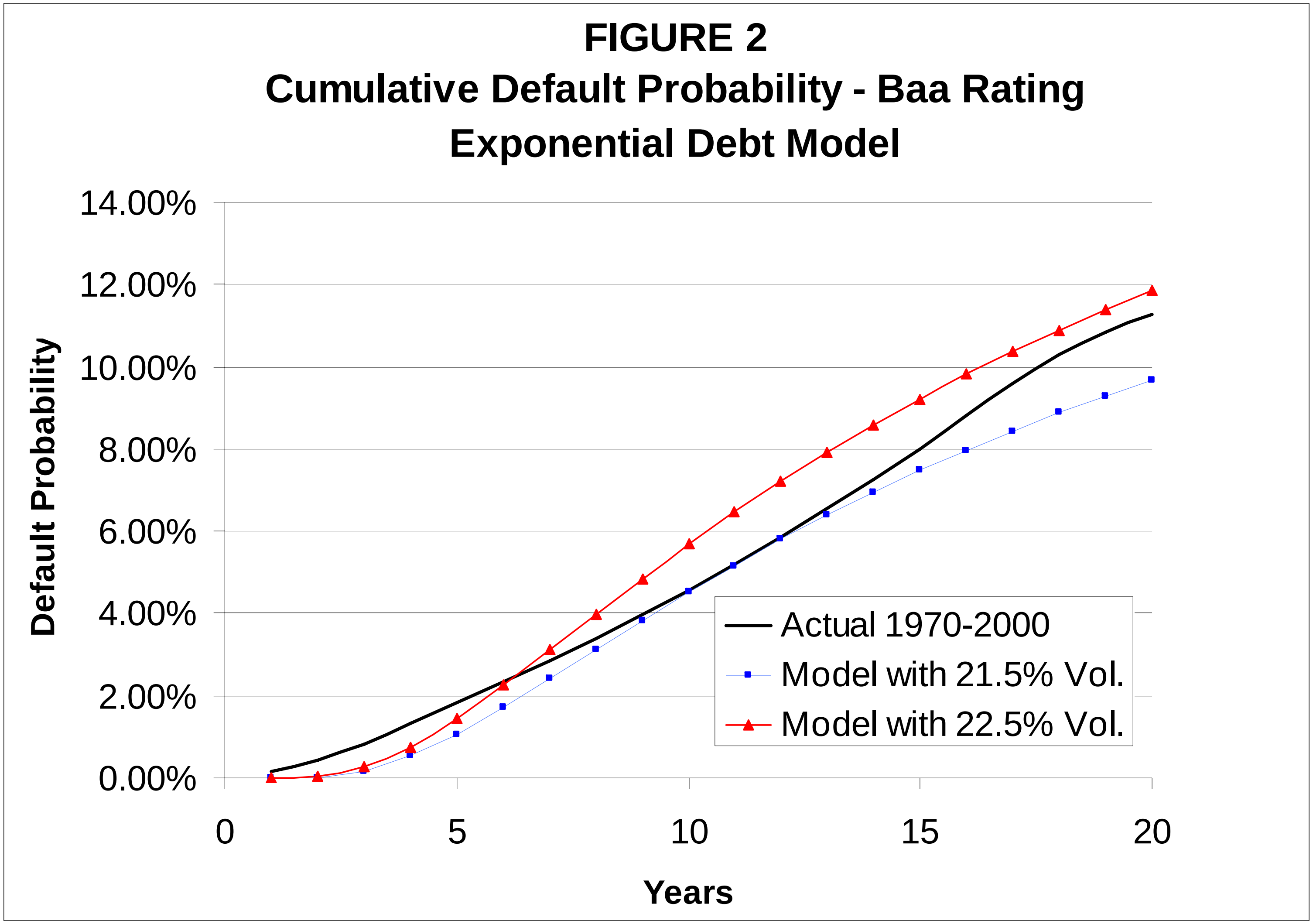
  ––This is very strange! For their parameters, quite similar to those here, I find *LT* ***underestimates spreads***. I can't replicate *EHH* results.

- ***A Possible Explanation (EG, HH): Spreads also reflect illiquidity***

  ––But Leland (JOIM, 2004) notes that ***probabilities of default should not be affected by bond market illiquidity***

  ––In contrast with bond market prices (and spreads)

Let's see if the model predicts ***cumulative default probabilities*** accurately:



- **For longer horizons (*t* > 7 yrs.), default probabilities OK:**

  are bounded above by model when σ = 22.5% and below when σ = 21.5%

  (Recall S&S estimate for Baa firms: σ = 22%)

. . .***But default probabilities are far too low at short horizons!***

(< 50% of actual when $t \leq 4$ yrs.)

- ***Observation: Even if illiquidity might explain too-low model spreads, it can't explain too-low short-term default predictions.***

- ***The Problem: a pure diffusion process for firm value!***

  ––Spreads and default rates → 0 as t → 0. (e.g. Lando (2004), pp. 14-15).

- ***A Possible Answer:*** ***Include jumps in asset value***

This is certainly not the first credit-risk model to consider jumps:

Credit risk (Zhou (2001), Duffie and Lando (2001), Hilberink & Rogers (2002), Giesecke & Goldberg (2003), H & H (2004), Chen & Kou (*CK*, 2005))

Regime changes (Hackbarth, Miao & Morellec (*HMM*, 2006))

- But most of these models are quite complex, and require numerical techniques to find solutions

- We consider a very simple mixed jump-diffusion process for asset value:

$$\frac{dV}{V} = (r - \delta + \lambda k)dt + \sigma\, dW \quad \textit{with probability}\ (1 - \lambda dt)$$
$$= -k \qquad \textit{with probability}\ \lambda dt, \quad 0 \le k \le 1$$

- Must adjust the ***drift*** of the diffusion to $g = r - \delta + \lambda k$

  to compensate for the jump, keep expected return rate = r - δ

- Adjust the ***volatility*** of the diffusion to $\sigma = (\sigma_L{}^2 - \lambda k^2)^{0.5}$

  (keeping long-horizon total volatility σ_L constant)

- A jump here represents a relatively rare "disaster",

  ––The firm loses a large fraction of its value and *liquidates* (Enron, Refco?)

  ––Note that unlike pure diffusion models, the recovery rate is random since *V* is random when a jump occurs

- Are jumps “rare”? Collin-Dufresne, Goldstein, Helwege (*CGH*, 2003):

“In practice, very few firms ‘jump’ to default. Indeed, since 1937, we are aware of only four firms that have defaulted on a bond which had an investment grade rating from Moody’s.”

––We don’t estimate the firm value process—just look at ***consequences if there were a rare jump*** on debt values, and default probabilities.

>> Observed default and recovery rates can be explained by an assumption of such jumps—*similar to “Dark matter”*??

Closed form solutions for Debt Value D

$$\boxed{D = \frac{C + mP}{z_1}\left(1 - \left(\frac{V}{V_B}\right)^{-y(g,z_1)}\right) + (1-\alpha)V_B\left(\frac{V}{V_B}\right)^{-y(g,z_1)} + \frac{\lambda(1-k)V}{z_2}\left(1 - \left(\frac{V}{V_B}\right)^{-y(g,z_2)}\right)} \quad (2)$$

where

$$g = r - \delta + \lambda k$$

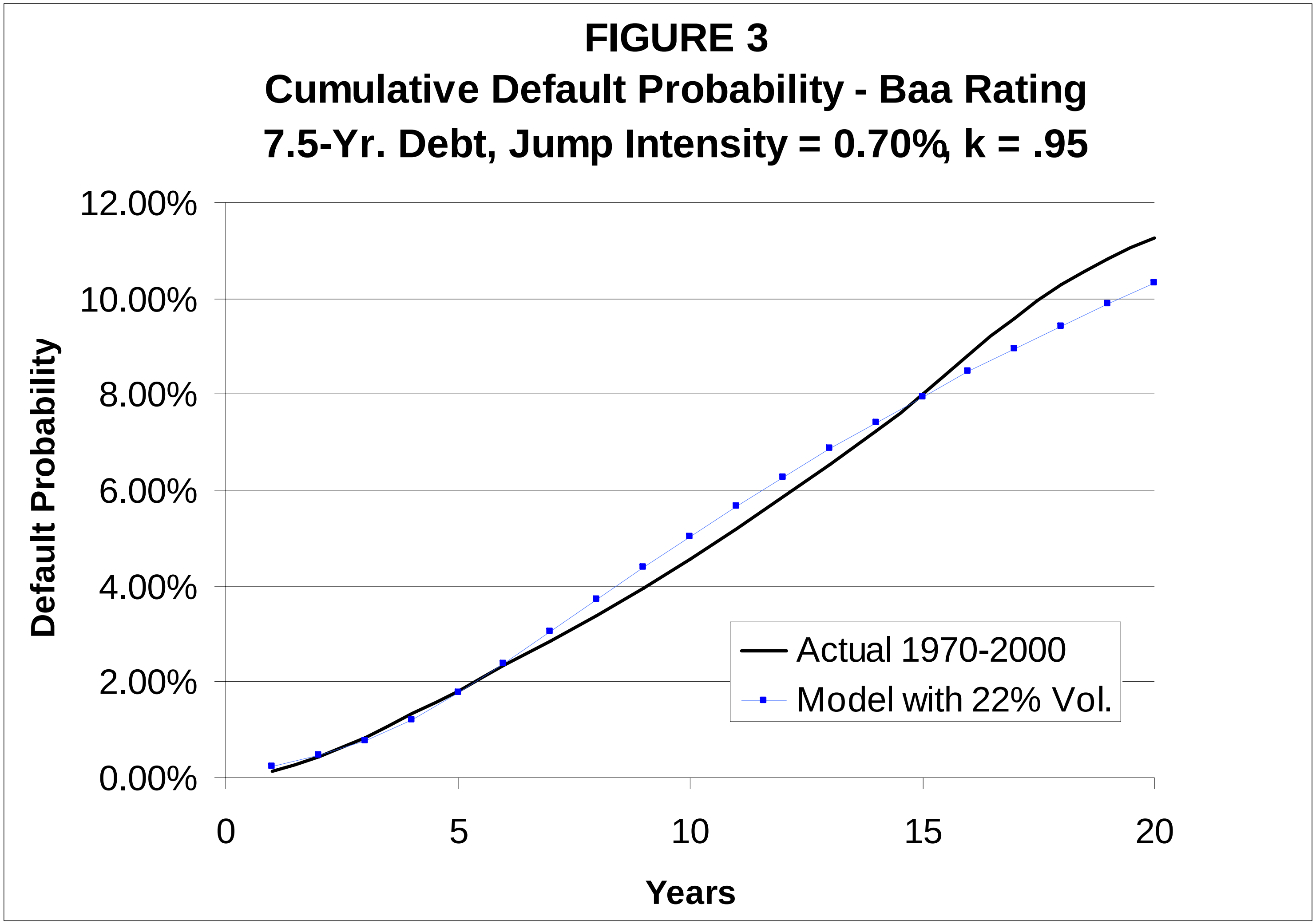
$$z_1 = r + m + \lambda$$

$$z_2 = z_1 - g$$

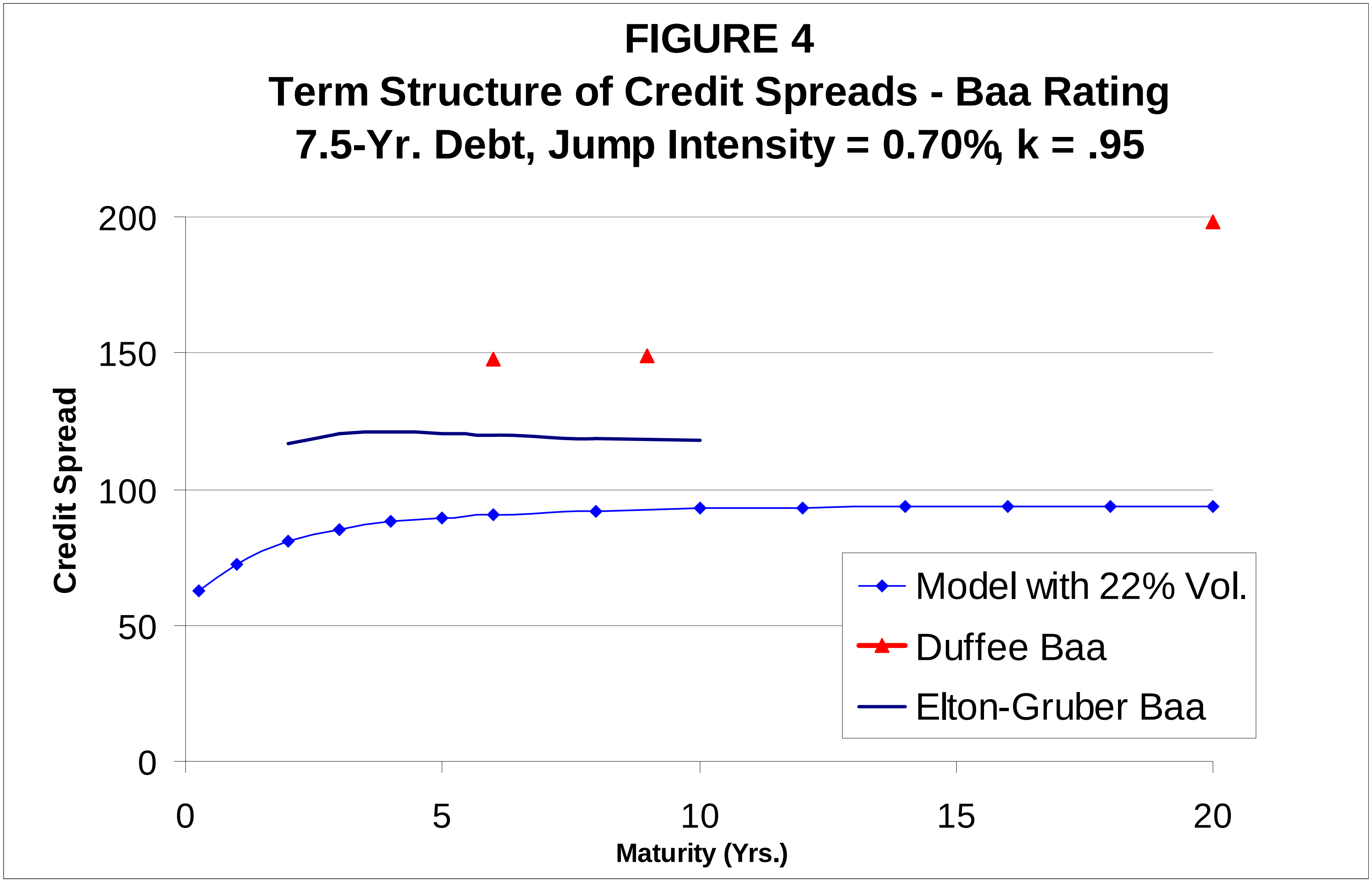
We also have closed-form solutions for V_B, E, and v.

Of course these formulas coincide with earlier formulas when $\lambda = 0$.

Predictions of Default at short horizons are now much better:



But predicted spreads are still too low:



LIQUIDITY

Longstaff, Mithal, Neis (2004): ***Find spreads for CDS are consistently lower than observed credit spreads***

- LMN attribute difference to non-default factors ("liquidity"), and find

  ––***The non-default component ranges from 50 to 72 bps per year, and "is nearly constant across rating categories."***

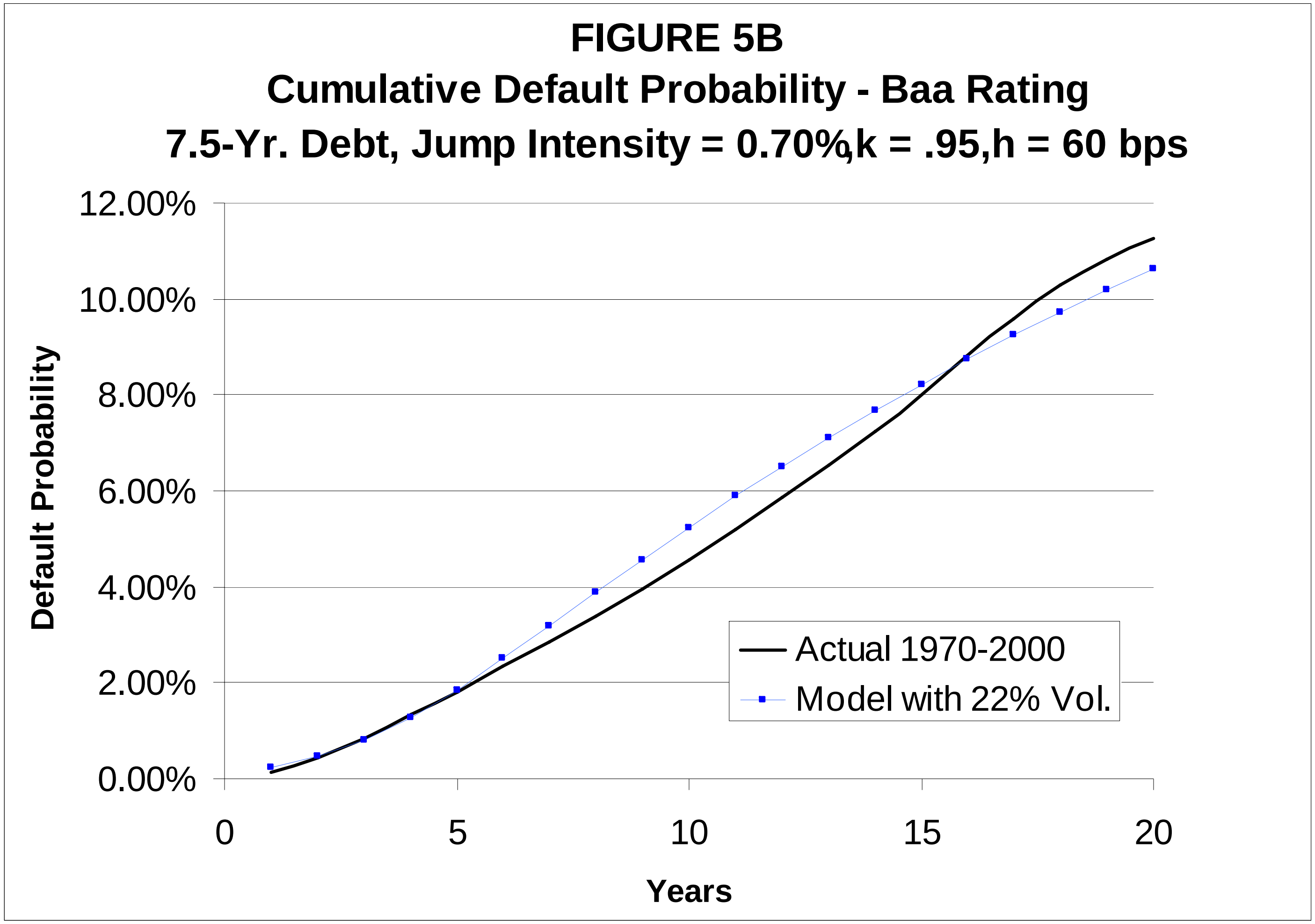
- We introduce the ***liquidity premium h (= 60 bps) as an addition to the required return on debt.*** (see also Ericsson & Renault (2005))

  >> That is, risk-neutral expected <u>debt</u> cash flows are discounted at r + h.

  >> Equity cash flows continue to be discounted at r.

  >> Not the same as just adding 60 bps to spread, since V_B will change.

Results with jumps, liquidity premium: Baa-rated debt



The model predicts a recovery rate of 49.5%, vs. the target of 50%.

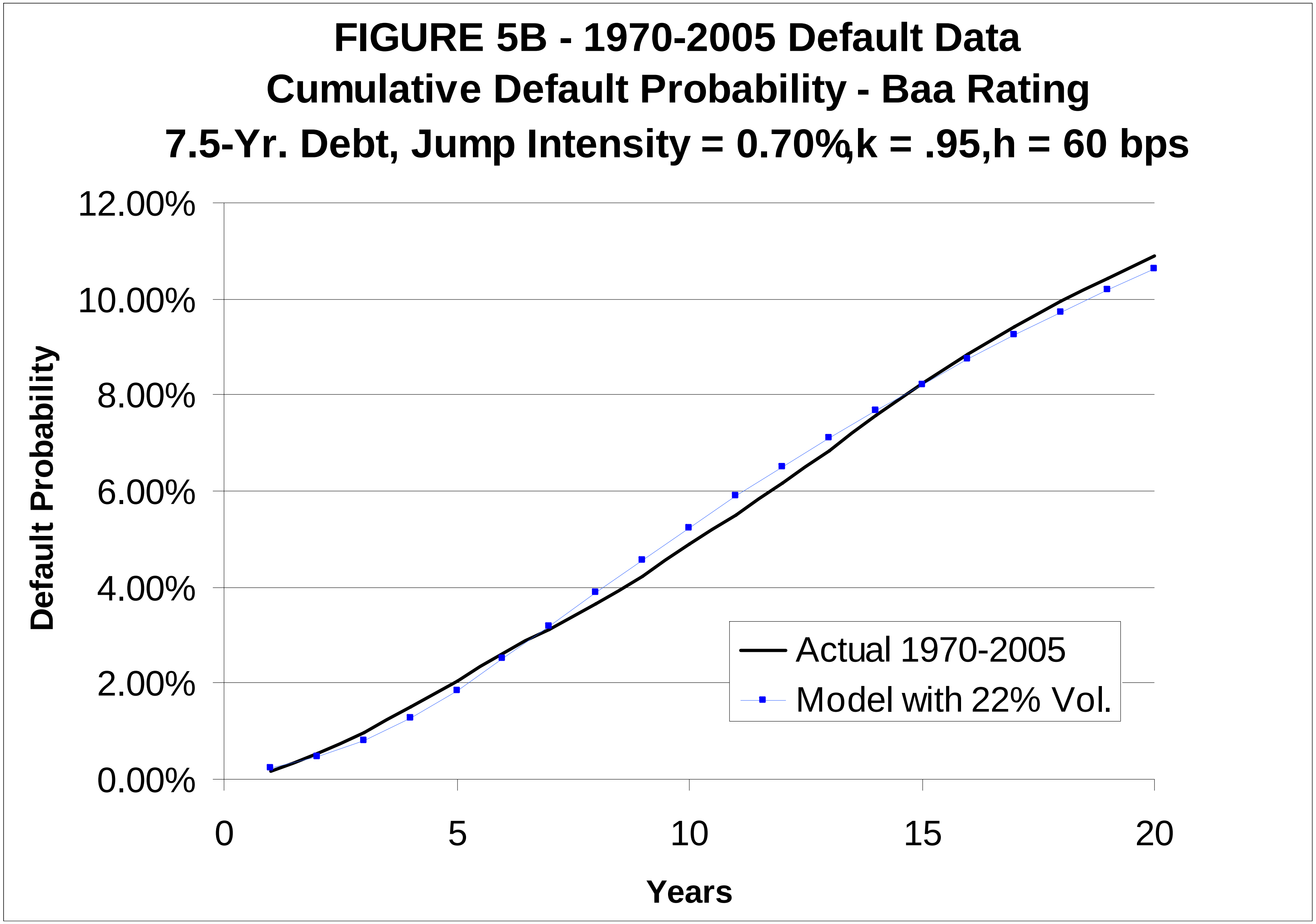


FIGURE 5B - 1970-2005 Default Data
Cumulative Default Probability - Baa Rating
7.5-Yr. Debt, Jump Intensity = 0.70%,k = .95,h = 60 bps
12.00%
10.00%
8.00%
6.00%
4.00%
2.00%
0.00%
Default Probability
0
5
10
15
20
Years
Actual 1970-2005
Model with 22% Vol.

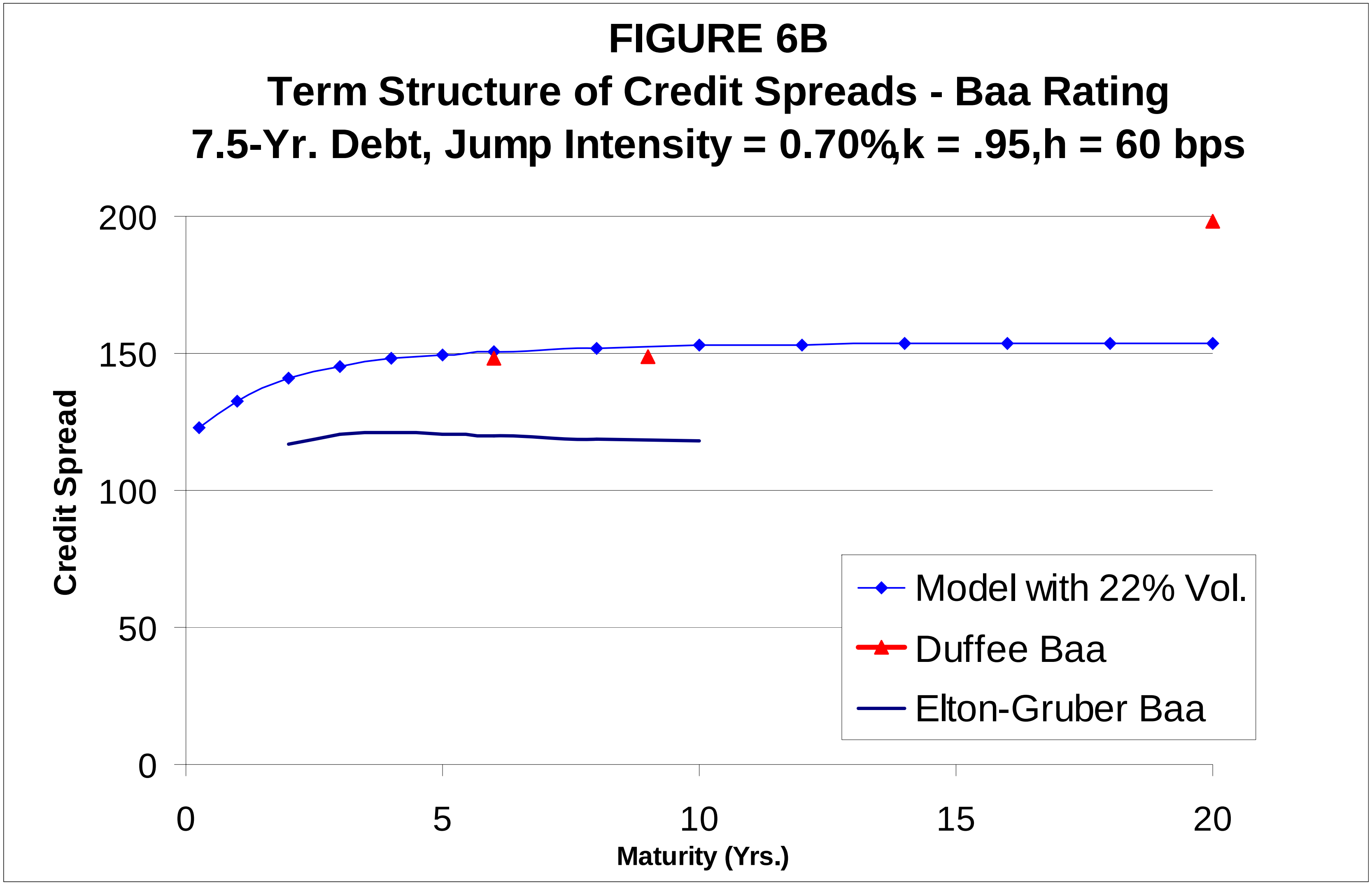
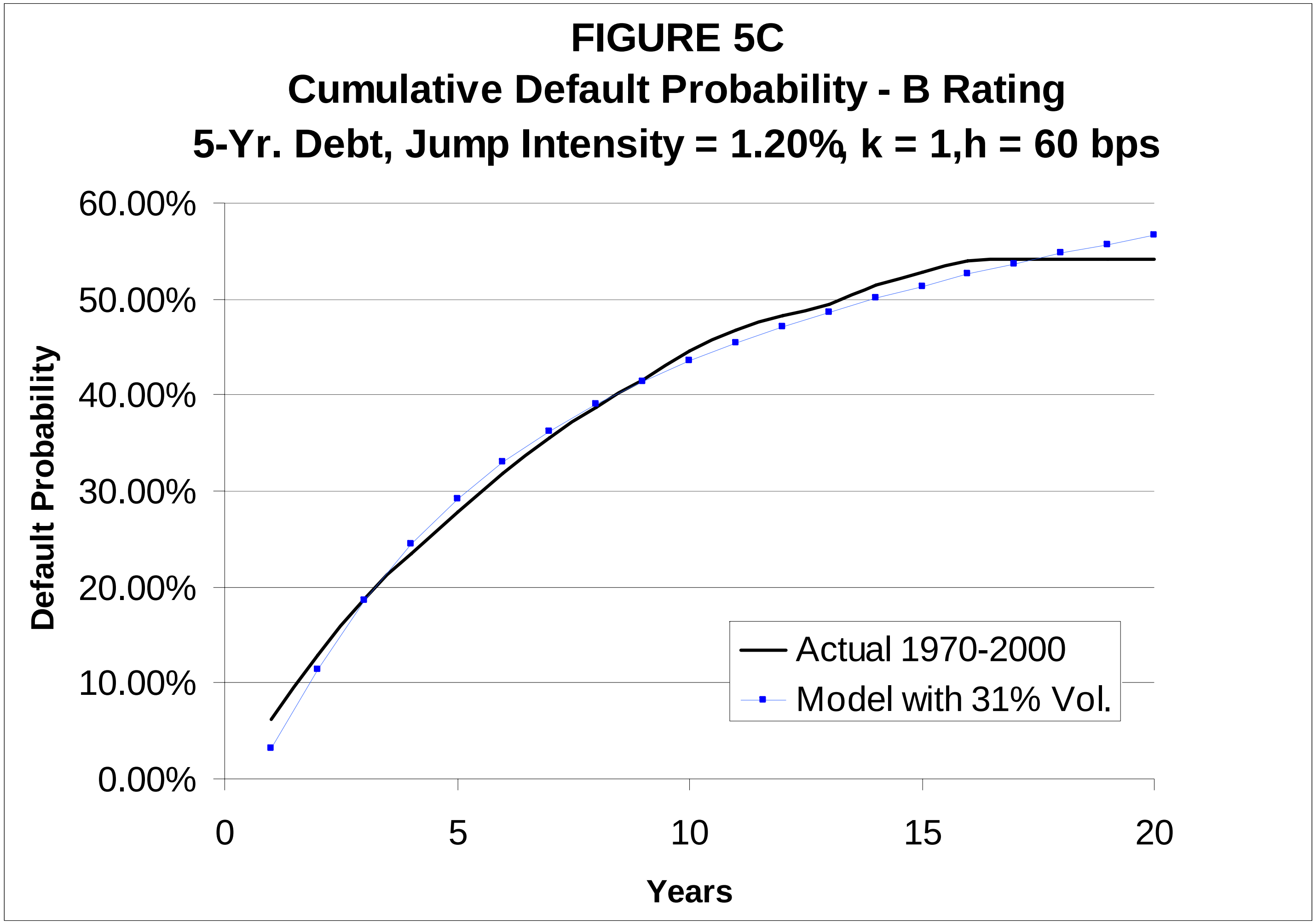


FIGURE 6B
Term Structure of Credit Spreads - Baa Rating
7.5-Yr. Debt, Jump Intensity = 0.70%,k = .95,h = 60 bps
Credit Spread
200
150
100
50
0
0
5
10
15
20
Maturity (Yrs.)
Model with 22% Vol.
Duffee Baa
Elton-Gruber Baa

Results with jumps, liquidity premium: B-rated debt



The model predicts a recovery rate of 40.5%, vs. the target of 40%.

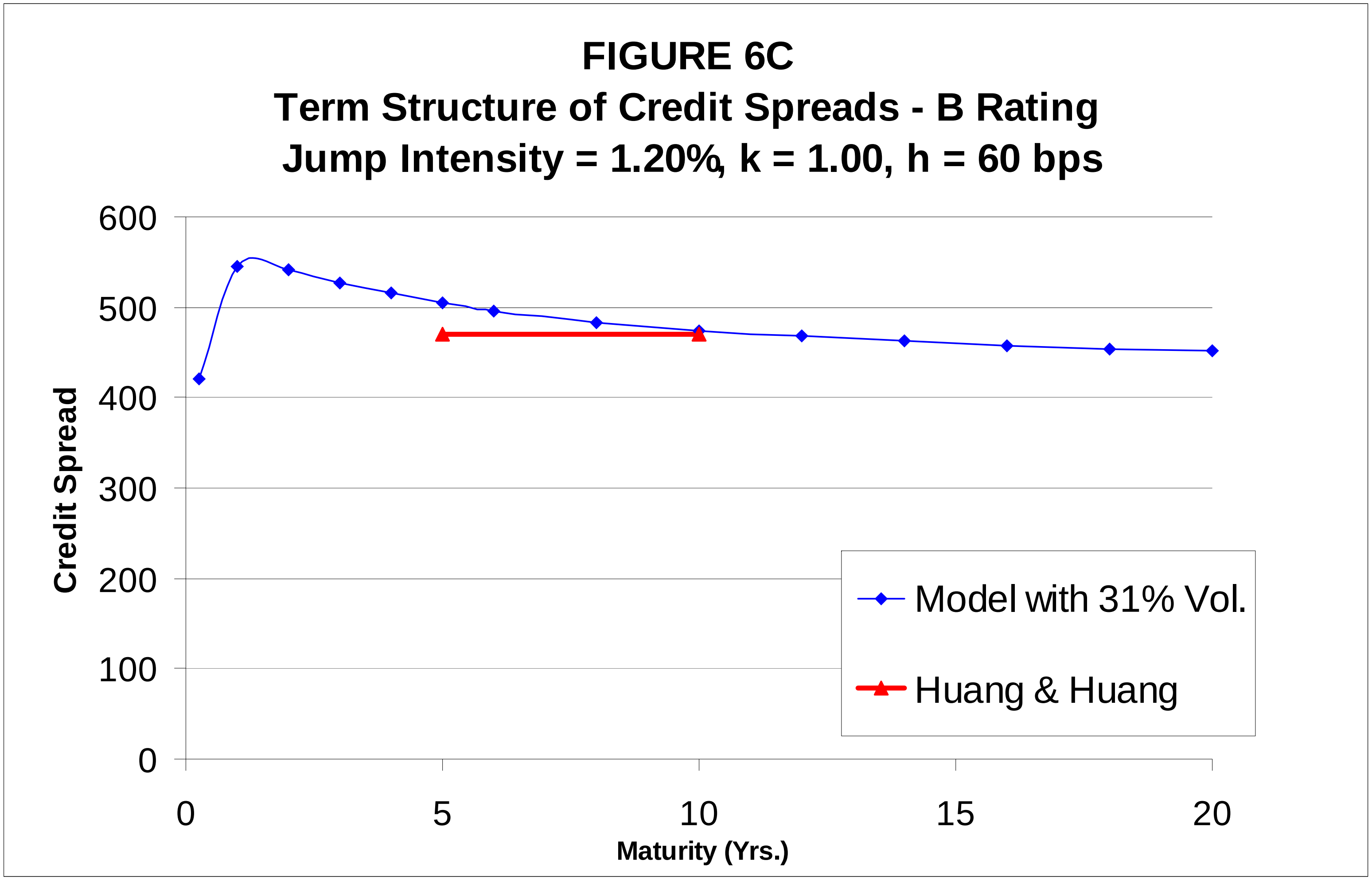
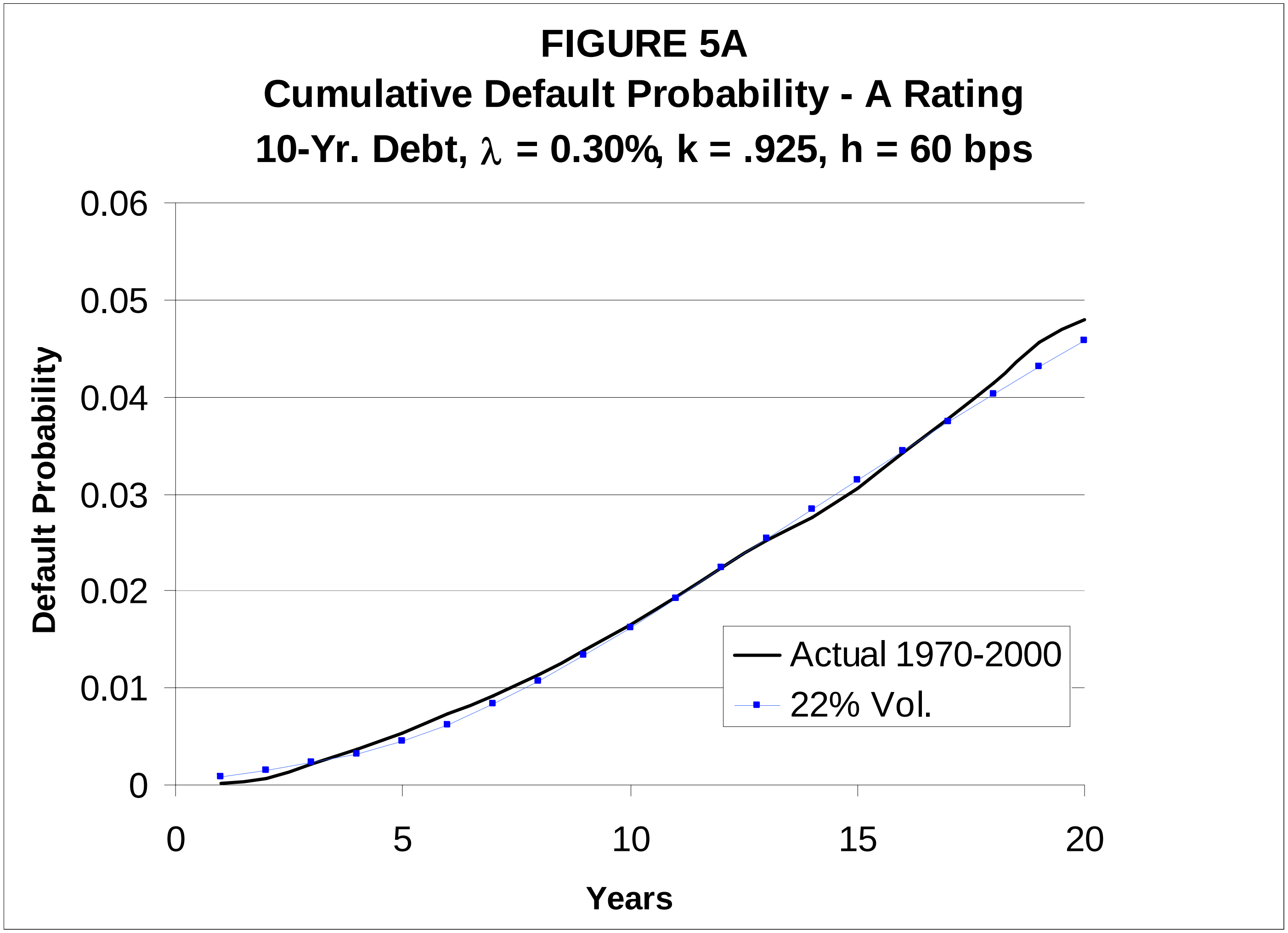


FIGURE 6C
Term Structure of Credit Spreads - B Rating
Jump Intensity = 1.20%, k = 1.00, h = 60 bps
600
500
400
300
200
100
0
Credit Spread
0
5
10
15
20
Maturity (Yrs.)
Model with 31% Vol.
Huang & Huang

Results with jumps, liquidity premium: A-rated debt



The model predicts a recovery rate of 59.5%, vs. the target of 60%.

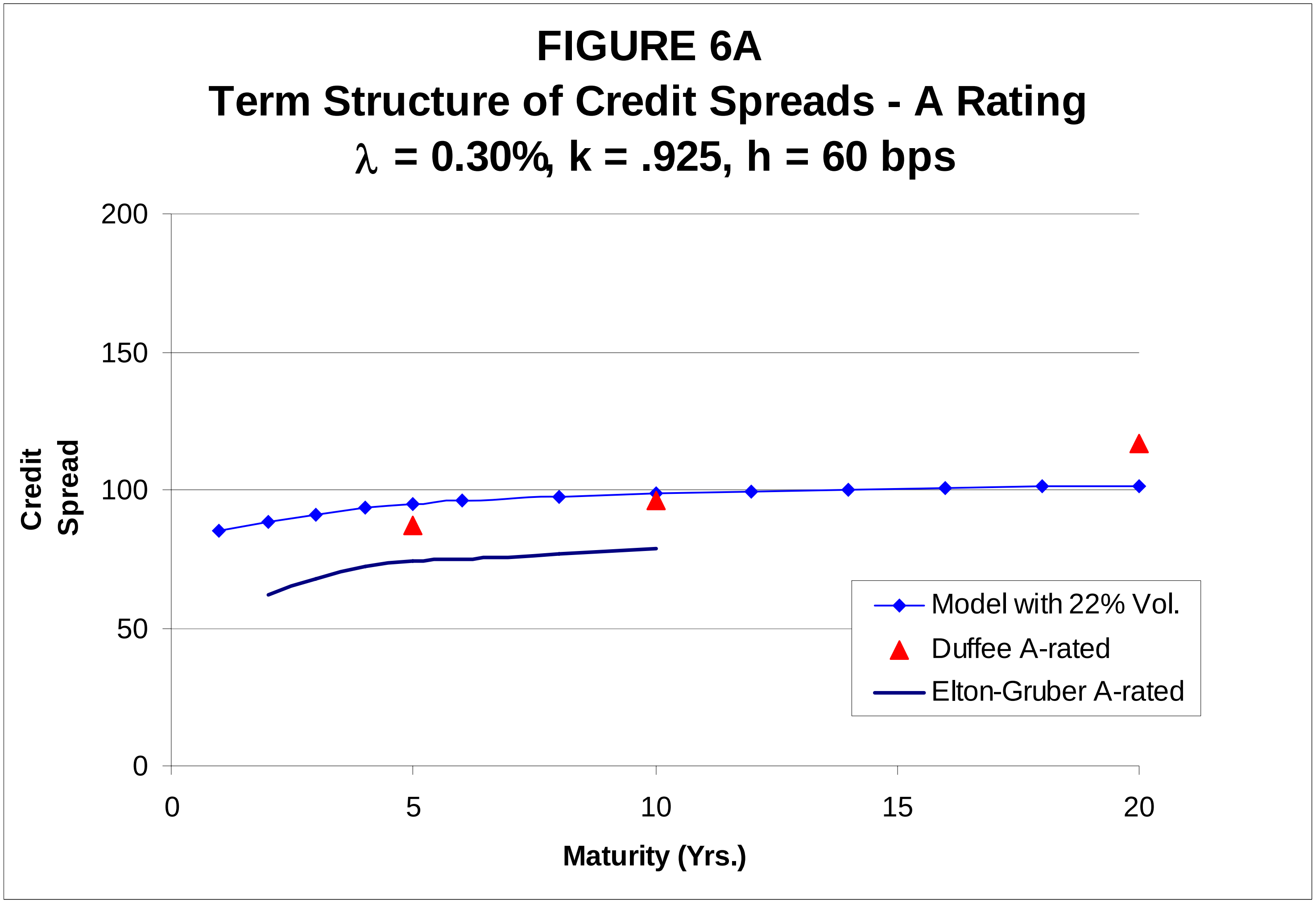


FIGURE 6A
Term Structure of Credit Spreads - A Rating
λ = 0.30%, k = .925, h = 60 bps
200
150
100
50
0
Credit Spread
0
5
10
15
20
Maturity (Yrs.)
Model with 22% Vol.
Duffee A-rated
Elton-Gruber A-rated

APPLICATIONS TO CORPORATE DECISIONS:

Optimal Capital Structure

- We now ***drop the assumption*** that leverage for firms matches the previously-specified levels (e.g. 43.3% for Baa-rated firms)

  ––We consider leverage ratios that ***maximize total firm value*** for firms in each different rating category.

- **Baa-rated firms:** Optimal leverage = **46.7%**

  ––This is not far from the actual ***Baa average leverage*** of **43.3%**

  ––If h = 0, optimal leverage is 49.9%.

- **A-rated firms:** Optimal leverage = **45.2%** (vs. actual 32.0%)

  >> A-rated firms appear to be somewhat ***under-leveraged***

  >> But the value loss is small (< 0.3% of firm value ***v***)

- **B-rated firms:** Optimal leverage = **36.7%**!! (vs. actual 65.7%)

  ––Less leverage than Baa because volatility higher, maturity 5 yrs.

  ––Spread at optimal leverage would be 240 bps, not 505 bps

  Tentative conclusion:

  ––Average B-rated firm in the data base is ***over-leveraged***

  ––Leverage stats for B-rated firm likely include ***fallen angels,*** whose initial leverage was lower

CONCLUSIONS

- ***Structural Models are alive and well!***

  ––With the addition of a simple jump and liquidity cost, they can explain both observed <u>credit spreads</u> and <u>default probabilities</u>

  ––Closed form solutions allow easy comparative statics

  ––Valuations can be used to study optimal financial structure of firms, as well as other corporate decisions

  ––Optimal leverage is close to actual leverage for Baa-rated firms

  >> A-rated firms appear to be under-leveraged relative to optimal
  >> B-rated firms appear to be considerably over-leveraged

APPENDIX

TARGETS: Following Huang & Huang (2004) and others; data 1985-1996

TABLE 1: TARGET SPREADS, DEFAULT DATA

Credit Spreads	***Targets***	***Sources***					
		Huang & Huang (HH, 2003), Duffee (1998), Elton & Gruber (EG, 2001)					
A Rated							
5 Yr.	**90 bps**	**HH:**	**96**	**Duffee:**	**87**	**EG:**	**74**
10 Yr.	**100 bps**	**HH:**	**123**	**Duffee:**	**96**	**EG:**	**79**
20 Yr.	**115 bps**	**HH:**	**N/A**	**Duffee:**	**117**	**EG:**	**N/A**
Baa Rated							
5 Yr.	**145 bps**	**HH:**	**158**	**Duffee:**	**149**	**EG:**	**121**
10 Yr.	**150 bps**	**HH:**	**194**	**Duffee:**	**148**	**EG:**	**118**
20 Yr.	**195 bps**	**HH:**	**N/A**	**Duffee:**	**198**	**EG:**	**N/A**
B Rated							
5 Yr.	**470 bps**	**HH:**	**470**	**(Based on Caouette, Altman, Narayanan (1998))**			
10 Yr.	**470 bps**	**HH:**	**470**	**(Based on Caouette, Altman, Narayanan (1998))**			
20 Yr.	**N/A**						
Riskfree Rate	**8%**	**HH:**	**8%**	**(Average over period 1985-1995)**			

Default Probabilities

Data: Moody's Special Comment 2001

A Rated		**Baa Rated**		**B Rated**	
1 Yr.	**0.01%**	**1 Yr.**	**0.14%**	**1 Yr.**	**6.16%**
5 Yr.	**0.54%**	**5 Yr.**	**1.82%**	**5 Yr.**	**27.90%**
10 Yr.	**1.65%**	**10 Yr.**	**4.56%**	**10 Yr.**	**44.60%**
20 Yr.	**4.79%**	**20 Yr.**	**11.27%**	**20 Yr.**	**54.20%**

Is there a jump risk premium?

––i.e., is there a difference between the risk neutral jump intensity $\boldsymbol{\lambda}$***,*** and the “real” (under the physical measure) intensity γ of a jump?

––***Yes***, if jump risk is imperfectly diversifiable.

––Measure by ratio ***H*** = γ/λ: smaller ratio ➔ larger jump risk premium.

––Given λ, the risk premium doesn’t affect *pricing* (spreads), but it must be known to determine the *probability of default* γ***.***

- *CGH* (2003) show that ***jump risk will command a risk premium*** if:

––Multiple firms can default simultaneously, or

––Default of one firm can increase default intensities of others.

––We assume a jump risk premium, but don’t know to need to know cause

Our approach: (alternative jump risk premia approaches are possible!)

- A jump to default is at least “as bad as” a diffusion to default, in that it should command at least as high a risk premium.

- We assume the *jump risk premium* ***H*** is the same as the *default risk premium* ***J*** *for the pure diffusion part of the asset value process*
- Let η be the cumulative default probability of the pure diffusion process at debt maturity using the *risk neutral* drift g , and

  ζ be the cumulative default probability of the pure diffusion process at debt maturity using the actual (physical) drift $(g + \pi)$, where

  π is the asset risk premium. Then the diffusion risk premium is

$$\boldsymbol{J} = \zeta \,/\, \eta < 1.$$

- For ***Baa*** debt, $\lambda = 0.70\%$ and $\pi = 4\%$/ yr. (see Lec.1 Table 2). After 10 yrs.,

  –– $\zeta = 1.84\%$, $\eta = 5.60\%$ ➔ $\boldsymbol{J} = \mathbf{.329}$

  –– Assuming $\boldsymbol{H} = \boldsymbol{J}$: Predicted real jump intensity $\gamma = \lambda^* \boldsymbol{J}$

  ➔ Real jump intensity $\gamma = 0.7\% \times .329 = 0.23\%$
- For ***B***-rated debt, $\lambda = 1.2\%$. At 5 yr. debt maturity, $\boldsymbol{J} = 25.6\%/35.1\% = \mathbf{.729}$

  ➔ Real jump intensity $\gamma = 0.88\%$

  ––If the jump risk premium is larger, default probabilities will be *lower*.