Implementing Allocations through the Tax System

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Abstract

We study how to implement an incentive compatible allocation in the market through a tax system in a setting where agents make multiple choices and are heterogeneous in multiple characteristics. An example shows that the canonical tax system proposed in Mirrlees (1976) is sometimes incapable of implementing the desired allocation. We derive necessary and sufficient conditions for a tax system to implement the allocation. These conditions can be used to check whether a proposed tax system implements the allocation. We show that such a check is not necessary if the allocation is bijective. In addition, we show that a monotonic or convex tax system can always implement the second-best provided i.) taxes are equated to wedges, ii.) the allocation is second-best to a welfarist planner and iii.) there are no externalities. The Mirrleesian implementation is effective if it meats these criteria, and it often does. Our work provides economists with a toolbox to design the optimal tax schedule. In addition, it sheds new light on sources of complexity in real-world tax and benefit systems.

Keywords: optimal non-linear taxation, redistribution, tax system, market implementation, price mechanism , private information

JEL-codes: H21, H22, D82, H24

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1 Introduction

In two seminal articles Mirrlees (1971, 1976) provides the foundation for the current literature on optimal redistributive non-linear taxation. In this model agents are heterogeneous in earnings ability. A social planner wants to redistribute from agents with high to agents with low earnings ability, but earnings ability is private information and hence the first-best is not available. The second-best allocation is derived in a direct mechanism. Each agent sends a message about his type to social planner. The social planner assigns bundles of choice variables to each agent based on the messages. The bundles that form the second-best allocation maximize the planner's welfare function subject to economic feasibility and incentive compatibility constraints. The constraints respectively ensure that the economy can produce the aggregate consumption, and that each agent weakly prefers his bundle over all other bundles in the allocation and thus truthfully reveals his type. This second-best allocation is characterized through the optimal wedges. These wedges define the difference between the marginal rate of substitution and the marginal rate of transformation for each type and each choice, and thus the optimal distortion from Laissez-Faire. Mirrlees proposed to implement this allocation through a non-linear tax system that satisfies two criteria. First, the optimal marginal tax should equal the optimal wedge. Second, the optimal tax system should be separable such that the marginal tax rate on a choice depends only on the consumption of the choice. This Mirrleesian implementation has become the canonical solution, and has subsequently been applied in many articles throughout the literature (see e.g. Atkinson and Stiglitz, 1976, Diamond, 1998, Saez, 2001, Bovenberg and Jacobs, 2005 and Golosov et al., 2013). Surprisingly, to the best of our knowledge, the general conditions under which this, or any, tax system implements the second-best allocation have never been derived.

In this paper we study the problem of implementation in more detail. In our the model agents differ in p, continuously distributed, unobserved characteristics such as earnings ability, patience and health status, and are assumed to maximize a utility function with multiple $(k+1) \geq p$ decision variables.[1] This allows us to study implementation in a very general setting. A large literature has been devoted to finding the allocation which maximizes a social welfare function subject to feasibility and incentive-compatibility constraints. We instead start off from an allocation that is assumed to be feasible and incentive-compatible and study the properties of tax systems that implement such an allocation through taxes.

We show that the class of tax systems which implement the allocation may

[1] Like most papers in public finance we assume that $k \geq p$ in order to apply the revelation principle (see Myerson (1979) and Townsend (1979)). A notable recent exception to this rule is Chone and Laroque (2010) who study the sign of the optimal wedge in the Mirrlees (1971) model under multi-dimensional heterogeneity with only one independent choice variable.

be limited, even though we have assumed that the initial allocation is incentive compatible. Intuitively, incentive compatibility requires that each agent prefers his bundle over the bundle of all other agents in a direct mechanism. However, in the market agents can create bundles which are not assigned in the direct mechanism. For example, an agent can create a budget-neutral joint deviation by increasing his consumption in one choice while decreasing it for an other. A tax system can only implement the allocation if such joint deviations are not optimal for any agent.

From the principle of taxation (derived by Hammond (1979), Rochet (1985), Guesnerie (1995) and Bierbrauer (2009)) we know that for every incentive compatible allocation at least one such non-linear tax system exists. In this tax system all joint deviations are prohibited, or taxed at a prohibitively high level. Since joint deviations are prohibited, incentive compatibility implies implementability. The resulting tax system can be very complex. In order to tax joint deviations prohibitively, the tax rate of each decision variable has to depend on the value of all decision variables. In addition, prohibitive taxes (or outright prohibitions) on a large part of the choice space seem unrealistic in real-world democracies. These undesirable qualities might explain why most papers in the literature propose to implement their allocation through a Mirrleesian tax system.

In this paper we present three results. First, we show through a simple counterexample that Mirrlees' tax system sometimes leads to outcomes far removed from the desired allocation because it allows for joint deviations. Second, we derive the necessary and sufficient conditions for a tax system to implement an allocation. Using standard micro-theory we show that taxes have to be equal to wedges, and indifference curves have to be more convex than budget constraints in all linear combinations of the decision variables. Economists can use these implementability constraints to verify whether a proposed tax system implements the desired allocation ex-post. That is, after formulating the entire tax-system the it can be checked whether the tax system satisfies these constraints.

Most optimal allocations in public finance do not have a closed-form solution. Solutions may be obtained through numerical simulations, but this implies that the ex-post check can only be performed on the special case that has been simulated. The ex-post check is useful in such simulations, but does not provide insights in the general properties of the optimal tax system. Our third result, however, identifies situations where any tax system that equates marginal taxes to wedges will implement the allocation. Two classes of problems can be distinguished. First, if there is a one-to-one correspondence between the type-space and the choice space incentive compatibility and implementability constraints coincide. Since we assumed the original allocation was incentive-compatible, implementability is ensured. In this case, equating taxes to wedges yields an unique tax system. Second, an allocation can be implemented by equating marginal taxes to optimal wedges if i.) the resulting tax system does not contain internal maxima, ii.) the allocation is second-best for a welfarist social planner, and iii.) there are no externalities. In practice most tax systems under consideration do not exhibit internal maxima because they are monotonic and/or convex over their entire domain. In addition, our current understanding of optimal non-linear redistributive taxation is mostly based on models where the planner is welfarist and there are no externalities.[2] In these models we can therefore derive general statements about the tax system by studying the wedges.

In this paper we make important methodological contributions. Our second result provides an ex-post check of implementability, while our third result shows that most problems studied in public finance have a relatively simple tax implementation. Our results yield additional insights into the complexity of implementation in other fields of mechanism design such as auction theory and monopoly pricing because we study the link between direct and indirect mechanisms. They indicate that a central planner that perfectly observes choices and can tax them non-linearly might still want to rely on legal prohibitions to reach the second-best allocation. We therefore provide a limit to the power of the price mechanism. This also has practical implications since it provides some intuition for the existence of (possibly optimal) complexities in the tax system in modern welfare states.

The rest of the paper is organized as follows. Section 2 discusses some literature on issues of joint deviation. Section 3 introduces the model and section 4 contains an example to illustrate the complexity of implementability. Section 5 provides the necessary and sufficient conditions that can be used as an ex-post verification of implementability. Section 6 uses these conditions to describe two classes of problems where implementability is guaranteed ex ante. Section 7 concludes.

2 Related literature

A large literature has been devoted to deriving the second-best allocation under multi-dimensional informational asymmetry (see e.g. Mirrlees, 1976, Armstrong, 1996, Armstrong and Rochet, 1999, Saez, 2002, Chone and Laroque (2010) and Renes and Zoutman, 2013). In addition, second-order incentive constraints have been studied in, for instance, Myerson (1981), Ebert (1992), Rochet and Chone (1998) and Hellwig (2007). The issue of how this second-best allocation can be implemented, that is, what the indirect mechanism looks like, is left often implicit in these papers. In this paper we do not attempt to derive any properties of the allocation, nor do we study the issue of incentive compatibility in the direct mechanism. Instead, we start from a given (first - and second-order) incentive-

[2]A notable exception is Jacobs and De Mooij (2011) which extends the Mirrlees model with externalities.

compatible and feasible allocation and study the tax system that implements such an allocation in the market. We show that incentive compatibility does not necessarily imply implementability, except in the sense of the restrictive tax system derived in Hammond (1979) that prohibits all joint deviations.

A particular class of joint deviations, known as unbalanced or skewed bidding, has been the studied extensively in the literature on procurement and auctions, and operations research management.[3] The literature focuses on a principal that needs to procure several goods in a single contract, but who has uncertainty over the exact quantities required at the time of the procurement procedure.[4] In the principal's first-best all goods are acquired from the cheapest firm at zero profit for the firm. However, the principal cannot observe the firm's cost structure. The principal therefore uses an reverse auction to select his supplier. The bidding firms state their price for each good and these are multiplied with score weights, yielding a scalar score. The firm with the lowest score wins. Using the expected quantities as score weights would seem to select the cheapest supplier. Unfortunately, if the expected quantities are slightly misestimated by the principal, the firm can create a profitable joint deviation. By asking more for the goods that are underweighted in the score rule and less for the overweighted goods, the bidder can increase expected payment while keeping his score constant. Even if the bidders do not wish to increase any price in isolation, such joint deviations are profitable. For risk neutral bidders the optimal bid contains infinite prices, yielding unbounded profit and risk. Renes (2011) studies mechanisms to prevent skewed bidding but finds no general solution when the principal is committed to accepting the bid with the lowest score. The author notes that legal rules in the US allow the government to reject unbalanced bids, creating a solution to the problem through prohibitions. Ewerhart and Fieseler (2003) study the optimal score-rule under uni-dimensional firm heterogeneity. Using the restriction that unit prices have to be weakly positive, and thus prohibiting a large part of the choice space of bidders, they recoup a version of the revelation principle and are able to determine a second-best allocation. These solutions are the logical equivalent of the principle of taxation applied to procurement. In both cases a large set of joint deviations or skewed bids are punished by giving them zero expected profit, and are thus effectively prohibited.

The New Dynamic Public Finance studies the Mirrlees' optimal taxation model in a dynamic setting where earnings ability follows a stochastic process.[5] Kocherlakota (2005), in line with the principle of taxation, shows that the optimal tax

[3]See for an overview Cattell et al. (2007) and Renes (2011).

[4]For simplicity we focus on procurement auctions in this review. However, these are simply reverse auctions and all issues encountered in procurement are also encountered in sale auctions. See Athey and Levin (2001) for an example of joint deviations in an sale auction.

[5]See Golosov et al. (2007) and Kocherlakota (2010) for an extensive overview of the literature.

system generally contains prohibitive tax rates on specific combinations of choices. Even in the case of an iid stochastic process Albanesi and Sleet (2006) find that excessive savings choices should be prohibited by for example setting a borrowing limit. This ensures the joint deviation of first saving too much and than working too little is not optimal. Because of these complex interdependencies in the tax-schedule implementation is very difficult. Farhi and Werning (2013) therefore investigate how close a tax-system with restricted tax-instruments comes to the second-best allocation in their simulations. We show that joint deviations can also occur in non-stochastic models. However, we show that there are important classes of problems where joint deviations are not an issue and prohibitive tax systems are not necessary. In future work we hope a similar classification can be applied to stochastic models.

3 The model

3.1 Agents' preferences

Our economy is populated by a unit mass of agents that are assumed to maximize a twice-differentiable utility-function:

$$u\left(\mathbf{x},y,\mathbf{n}\right) \tag{1}$$

where $\mathbf{x} \in \mathbf{X} \subseteq \mathcal{R}^k$ is a set of k decision variables, or goods. The numeraire good $y \in Y \subseteq \mathcal{R}$ s assumed to be an untaxed normal good, $u_y > 0, u_{yy} \leq 0$. Decision variables $\mathbf{x}$ and y are observable at the individual level and the social planner can tax the goods in $\mathbf{x}$ through a fully non-linear tax system. Throughout the rest of the paper we will sometimes refer to the decision variables in $\{\mathbf{x}, y\}$ as goods, even though they can be both inputs and outputs to the production process.

Agents are heterogeneous in a p-vector of unobserved characteristics, $\mathbf{n} \in \mathbf{N} \subseteq \mathcal{R}^p$. Characteristics in $\mathbf{n}$ could be for example earnings ability, patience and health status. We will refer to an element in $\mathbf{n}$ as a characteristic of the agent and to a specific vector $\mathbf{n}$ as the type of the agent. We assume $\mathbf{n}$ follows a multidimensional differentiable cumulative distribution function $F\left(\mathbf{n}\right)$, with $F : \mathbf{N} \to [0, 1]$ and probability density $f\left(\mathbf{n}\right) \geq 0 \quad \forall n \in \mathbf{N}$. For simplicity we assume the inequality holds strictly on the interior of $\mathbf{N}$.[6]

Our model may be either static or dynamic, depending on whether decisions in $\mathbf{x}$ occur in the same period or in different periods. However, we do assume that all characteristics in $\mathbf{n}$ are revealed before the first period.

[6]See Hellwig (2010) for a treatment of uni-dimensional incentive problems where the type-distribution may have holes and mass points.

It is assumed that $k \geq p \geq 1$, such that there are at least as many decision variables in $\mathbf{x}$ as characteristics in $\mathbf{n}$. This allows the application of the revelation principle in our analysis (see Myerson, 1979 and Townsend, 1979) since it guarantees that the choice-space is large enough to reveal all relevant characteristics.

Preferences can be summarized by:

$$\mathbf{s}(\mathbf{x}, y, \mathbf{n}) \equiv -\frac{u_{\mathbf{x}}(\mathbf{x}, y, \mathbf{n})}{u_y(\mathbf{x}, y, \mathbf{n})}.$$

$\mathbf{s}$ denotes the vector of shadow prices. Element s_i is the (negative) relative preference for decision variable x_i with respect to the numeraire y. Therefore, s_i represents the marginal loss of receiving an extra unit of x_i, expressed in units of the numeraire variable y.

3.2 Incentive compatibility and feasibility

Each agent sends a p-dimensional message about their type to the agent. The central planner sends bundles of goods to the agent on the basis of the p-dimensional message. This allocation of goods is denoted by:

$$\{\mathbf{x}^*(\mathbf{m}), y^*(\mathbf{m})\} \quad \forall \mathbf{m} \in \mathbf{N}.$$

Here, $\mathbf{m}$ is the p-vector message each agent sends to the planner. The function $\mathbf{x}^*$ maps from the message space to the good space, $\mathbf{x}^* : \mathbf{N} \rightarrow \mathbf{X}$ and y^* maps from the message space to the numeraire good space, $y^* : \mathbf{N} \rightarrow Y$. We assume $\mathbf{x}^*(.)$ and $y^*(.)$ are both twice differentiable in all their arguments.

An allocation has to satisfy the economies resource constraint in order to be feasible. We assume the economies resource constraint takes the form:

$$\int y dF(\mathbf{n}) + R = \int q(\mathbf{x}) dF(\mathbf{n}) \tag{2}$$

In this equation, R is the exogenous revenue requirement of the government and $q(.)$ is the economies production function. The equation states that total production should the sum of consumption of the numeraire and exogenous government expenditure. Derivatives q_{x_i} may be positive or negative depending on whether x_i is an input or an output of the production process. We assume weakly decreasing returns to scale such that all $q_{x_i x_i}$ are non-positive.

Each agent sends the message that maximizes his utility. Individuals can send a message which truthfully reveals their type $\mathbf{m} = \mathbf{n}$ or they can mimic a different type by sending a message $\mathbf{m} \neq \mathbf{n}$.

Definition 1 *An allocation* $\{\mathbf{x} = \mathbf{x}^*(\mathbf{n}), y = y^*(\mathbf{n})\}$ $\forall \mathbf{n} \in \mathbf{N}$ *is incentive compatible and feasible if each agent truthfully reveals his entire type in a direct mechanism:*

$$\mathbf{n} = \arg\max_{\mathbf{m}} u(\mathbf{x}^*(\mathbf{m}), y^*(\mathbf{m}), \mathbf{n}) \quad \forall \mathbf{n} \in \mathbf{N}. \tag{3}$$

and in addition satisfies equation (2).

Incentive compatibility requires each agent to weakly prefer his bundle over the bundle designed for any other agent. As such, agents truthfully reveal their type in the message they send to the planner. Conditions under which this condition holds are derived in a.o. Mirrlees (1976) and McAfee and McMillan (1988).

The optimal distortion in the economy can be characterized by the wedges each agent faces on the allocation:

$$\mathcal{W}_i(\mathbf{n}) = s_i(\mathbf{x}^*(\mathbf{n}), y^*(\mathbf{n}), \mathbf{n}) - q_{x_i}(\mathbf{x}^*(\mathbf{n})). \tag{4}$$

These wedges represent the difference between the marginal rate of substitution of the agent and the marginal rate of transformation (or production price) on the allocation. If the wedge is positive a good in the allocation is distorted below its Laissez-Faire value (i.e taxed) and vice versa.

3.3 Market implementation

An incentive-compatible and feasible allocation is the starting point of our analysis. The aim of this study is to find properties of a tax system that implements such an allocation in the market. Therefore, we have to go beyond the direct mechanism and study the choice problem of agents in a market. Agents maximize their utility function (1) subject to their budget constraint in the market:

$$y \leq q(\mathbf{x}) - T(\mathbf{x}), \tag{5}$$

such that how much a consumer can spend on y depends on his choice of $\mathbf{x}$, the production function $q(.)$ and the tax system, $T(.)$. We assume the tax system is twice differentiable, but we do not put any other a priori restriction on the tax function. The tax function may be fully non-linear and can contain arbitrarily complex interdependencies.

By Walras' law if the economies resource constraint, (2), and the agents' budget constraints, (5), are simultaneously satisfied, the government's budget constraint must also be satisfied. Therefore, if the tax system is successful in implementing a feasible allocation, we do not need to check whether the government budget is balanced.

A tax system implements an allocation if each agent weakly prefers his bundle over all other combinations of goods available to him in the market. This concept is formally defined in definition 2:

Definition 2 *A tax system implements an allocation* $\{\mathbf{x}^*(\mathbf{n}), y^*(\mathbf{n})\}$ *if each agent selects the bundle on the market that was assigned to him in the allocation:*

$$\begin{aligned} \{\mathbf{x}(\mathbf{n}), y(\mathbf{n})\} &= \max_{\mathbf{x},y} \{u(\mathbf{x},y,\mathbf{n}) : y = q(\mathbf{x}) - T(\mathbf{x}), \mathbf{x} \in \mathbf{X}, y \in Y\} \\ \forall \quad \mathbf{n} &\in \mathbf{N} \end{aligned} \tag{6}$$

The difficulty of implementability can be understood by comparing definitions 1 and 2. In the direct mechanism the agent maximizes his utility by sending out the optimal p-dimensional message. On the market the agent maximizes his utility by choosing k goods. Since, $k \geq p$ the agent can deviate in more directions on the market than in the direct mechanism. That is, the market allows the agents to create new bundles that were not assigned in the direct mechanism. Such a strategy is a joint deviation, because in order to create a new bundle while simultaneously satisfying the budget constraint an agent has to deviate in at least two goods. A tax system satisfies definition 2 only if such joint deviation strategies are never optimal.

Mirrlees (1976, sect. 3) defines the canonical tax implementation in the case of $p = 1$. The Mirrleesian implementation has two properties. First, taxes are equated to wedges such that $T_i'(\mathbf{x}^*(n)) = \mathcal{W}_i(n)$ for all goods. That is, the marginal tax on a good is equal to the optimal wedge of the good. This is a very intuitive property for a tax system since it sets the market price of the good equal to the individuals' shadow price. In section 5 we will show that this is always a property of an optimal tax system. Second, $T_i'(\mathbf{x}^*(n))$ equals $T_i'(x_i(n))$. That is, the optimal tax on good i depends only on the consumption of good i and the tax is separable. This property makes taxing joint deviations at prohibitive levels impossible, since interdependencies are required to identify joint deviations. As a result it is unclear if and when the Mirrleesian implementation achieves the desired allocation. In the next section we show through a counter-example that incentive compatibility is not sufficient to guarantee implementability through a Mirrleesian tax schedule.

4 Mirrleesian implementation: A counter-example

Figures 1 and 2 give a clear example of a case where a Mirrleesian implementation does not achieve the desired result.[7] The figures describe a situation with two goods in $\mathbf{x}$ and one exogenous characteristic, $k = 2$ and $p = 1$. The optimal allocation assigns one bundle $\{x_1^*(n), x_2^*(n), y^*(n)\}$ to each type. Since there is only one hidden characteristic, this allocation forms a line in $X_1 \times X_2 \times Y$ space. This line is represented by the black lines in figures 1 and 2. The dots represent the bundle of one particular type. The hyper-plane in the figures shows the budget

[7] The mathematics behind this example can be found in the appendix.

constraint of individuals in the unique Mirrleesian implementation where taxes are equated to wedges and the tax system is separable.

Figure 2 represents the utility function of the agent with the assigned bundle at the dot, for all combinations $\{x_1, x_2, y\}$ that satisfy his budget constraint with equality in this implementation. In figure 2 we can see that the assigned bundle (dot) marks the highest utility level on the allocation (line), such that the agent prefers his bundle over any of the other bundles in the allocation. The allocation is therefore incentive compatible for this type and has to be either a maximum or a saddle-point in utility.

However, in the market the agent can deviate from the allocation (line) by creating a joint deviation. In this example joint deviations yield much higher utility. So although the allocation is incentive compatible, the Mirrleesian implementation does not implement it.

The fact that Mirrlees' tax implementation is not effective in this example does not mean that the allocation cannot be implemented through taxes. Clearly, the government can use non-price tools like legal rules to prohibit all choices outside of the allocation, thereby decreasing the effective choice set of the agent. Equivalently the government could levy prohibitively high tax rates on the off-allocation choices to the same effect. In this sense, the need to use interdependencies with prohibitive taxes signals that the limits of the price mechanism as a means to influence behavior are reached and implementation becomes very restrictive. In some cases there may be less restrictive implementations. The next section derives necessary and sufficient conditions for a tax-system to implement the second-best.

5 Conditions for implementability: an ex-post check

Proposition 1 derives the general conditions under which a differentiable tax system implements an allocation, by formally solving the problem of definition 2. Note that we restrict ourselves to differentiable tax systems. As such, it is technically impossible to prohibit joint deviations in the spirit of Hammond (1979). However, it is possible to raise the marginal tax rate for joint deviations to arbitrary high levels. Since the utility function is also differentiable, such a marginal tax rate acts as a *de facto* prohibition.

Proposition 1 *An incentive compatible and feasible allocation can be implemented through a twice differentiable tax system* $T(\mathbf{x})$ *iff* $\forall \mathbf{n} \in \mathbf{N}$*:*
i.)

$$y^*(\mathbf{n}) = q(\mathbf{x}^*(\mathbf{n})) - T(\mathbf{x}^*(\mathbf{n})), \tag{7}$$

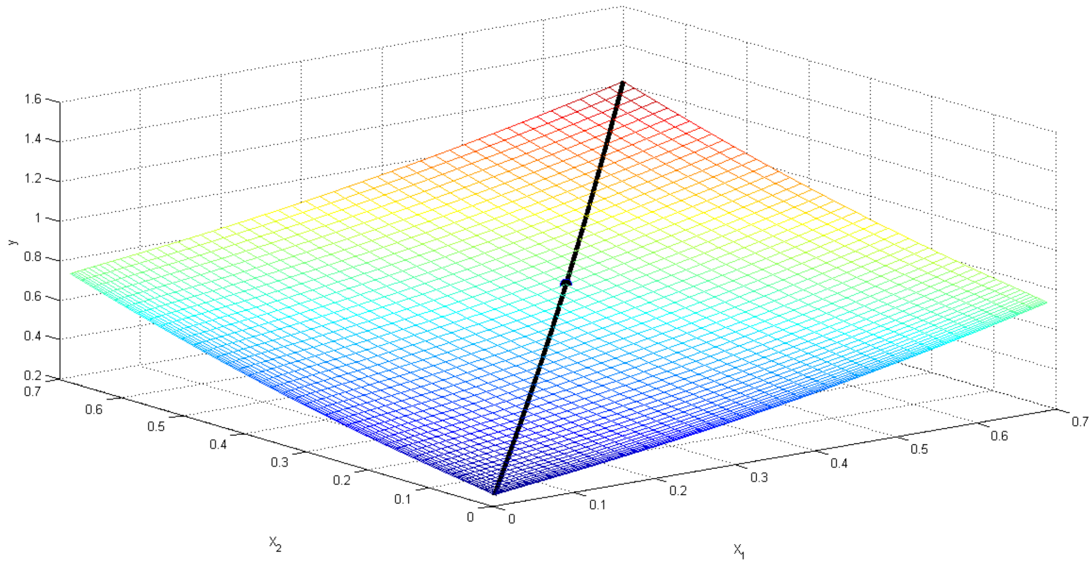

Figure 1: An optimal allocation and a budget constraint.

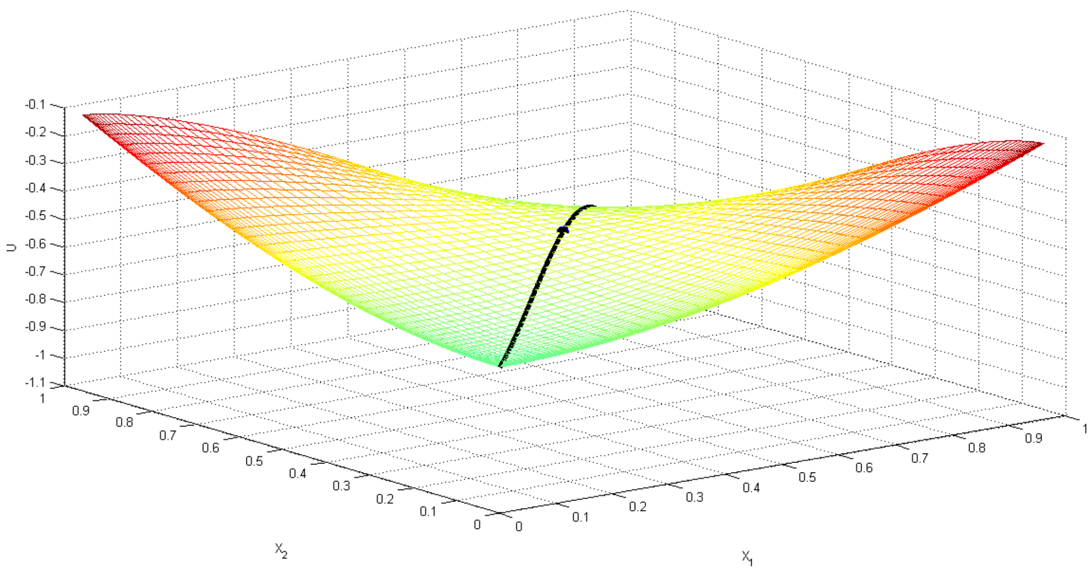

Figure 2: Utility of an agent faced with the budget constraint in Figure 1.

ii.)

$$\mathcal{W}(\mathbf{n})_i = T_i' \left(\mathbf{x}^* \left(\mathbf{n}\right)\right), \tag{8}$$

iii.)

$$-\frac{\partial \mathbf{s}\left(\mathbf{x}, y\left(u, \mathbf{x}, \mathbf{n}\right), \mathbf{n}\right)}{\partial \mathbf{x}} + q''\left(\mathbf{x}^*\left(\mathbf{n}\right)\right) - T''\left(\mathbf{x}^*\left(\mathbf{n}\right)\right) \lessdot 0. \tag{9}$$

Proof. The proof can be found in the appendix. ■

Equation (7) ensures that the amount of taxes paid for any bundle of $\mathbf{x}^*(\mathbf{n})$ within the allocation is uniquely determined. If the total tax level $T(\mathbf{x}^*(\mathbf{n}))$ is too high, the tax schedule cannot implement the allocation because people receive too little $y^*(\mathbf{n})$ if they choose their assigned quantities $\mathbf{x}$, and vice versa. Equation (8) is the first order condition for a market implementation. It states that taxes are equated to wedges. There are always as many marginal tax rates in T' as there are goods in $\mathbf{X}$, for all $\mathbf{n} \in \mathbf{N}$, such that there is always a unique vector of marginal tax rates $T'(\mathbf{x}^*(\mathbf{n}))$ that satisfies (8) within any possible incentive compatible allocation. In effect this means that the first order conditions of this problem can always be met and that the solution is unique on the allocation.

Equation (9) states that the indifference curve of any linear combination of $\mathbf{x}$'s with respect to y should be more convex than the budget constraint for the same linear combination of $\mathbf{x}$. This condition is different from the standard condition of utility maximization with two goods (see e.g. Mas-Collel et al., 1995) in two ways. First, in standard micro-economic theory the budget constraint is linear and hence if the indifference curve is convex, it is automatically more convex than the budget constraint. Here, the budget constraint is non-linear and therefore sufficiency requires the indifference curve to be more convex than the budget constraint. Second, since there are multiple choices, sufficiency requires that the indifference curve of all linear combinations of $\mathbf{x}$ with respect to y are more convex than the budget-constraint.

Since the conditions derived are both necessary and sufficient, they can be used to verify whether or not a specific tax system implements an allocation.

6 Sufficient Ex-Ante Conditions for Implementability

The necessary and sufficient conditions of proposition 1 can be a useful test to verify that a specific tax-system implements a specific allocation. However, in order to perform the check, one needs to derive the entire allocation. In many cases an explicit solution for the allocation that we are interested in is not available. Numerical solutions are available, but these. by definition, describe only special cases. It would therefore be useful to classify problems where the second-order implementability constraints (9) are never violated. For such problems, any tax system where marginal tax rates are equated to wedges implement the desired allocation. As such, the optimal tax system can be fully characterized by studying the optimal wedges and we make more general statements about tax systems. In this section we show that there are at least two such cases.

6.1 A bijective allocation

The combination of (7) and (8) defines the tax schedule on the allocation. If the allocation perfectly covers the choices space this tax-schedule must implement the allocation. Since every choice corresponds to the choice of a type, and every type prefers his bundle over the bundle of the other types, incentive compatibility and implementability coincide. Proposition 2 provides a sufficient condition for such a unique tax implementation to exist.

Proposition 2 *The tax implementation is unique if the mapping* $\mathbf{x}^*(\mathbf{n})$ *is bijective*

Proof. proof in appendix ■

The allocation derived in Mirrlees (1971) is an example of a bijective allocation, provided the ability distribution is unbounded. In the direct mechanism all ability types are assigned a specific gross income level x. Mirrlees shows that if ability is continuously distributed in $\mathcal{R}_+$, the second-best allocation assigns all gross income levels to a specific ability type without bunching. Hence, the function $x^*(n)$, mapping ability to gross income, is bijective. It follows that the entire tax system is determined by (7) and (8). The principle of taxation Hammond (1979) extends this idea. If the allocation is bijective incentive compatibility and implementability coincide. If all joint deviations are prohibited the allocation must be bijective in the effective choice space. Hence, implementability is guaranteed. However, as we have noted before, this restriction of the choice-space might result in a rather unrealistic tax system.

6.2 Second-best of a welfarist planner

As we have seen in figure 2 a tax system where taxes are equated to wedges may sometimes place agents on utility saddle-points, with the result that they will surely deviate from the desired allocation. In the next proposition we show that an allocation that places individuals in saddle-points allows Pareto-improvements and thus cannot be the optimum of a welfarist planner if there are no externalities.

Proposition 3 *If an allocation maximizes a welfare function* $SW = \int W(u)\, dF(\mathbf{n})$, *subject to the incentive compatibility constraints and the resource constraint, and*

$W' \geq 0$, then any tax $T(\mathbf{x})$ schedule that does not have an internal maximum in any linear combination of the $\mathbf{x}$'s on the allocation and satisfies (7) and (8) has to satisfy (9).

Proof. The proof can be found in the appendix. ■

Intuitively, any tax schedule that does not satisfy (9) allows for Pareto improvements and cannot be second best. By definition there exists at least one deviation that increases utility for at least one agent if (9) is not met for all agents. In addition, since the first-order conditions (8) combined with the violation of (9) imply that the agent is located in a saddle-point, the exact opposite deviation must increase his utility by approximately the same amount. This can easily be seen in figure 2. The agents' utility increases as much if he moves to the right off the allocation as when he moves to the left. Provided tax revenue is not maximized in the allocation, tax revenue must weakly increase either for the deviation to the right, or for the deviation to the left. In figure 1 the tax schedule is monotone and hence such a deviation exists. Therefore, the allocation in figure 1 could not have been second-best for a welfarist social planner.

This proof breaks down in the presence of externalities. The deviation of any agent can influence the utility of other agents through the externality, such that it is unclear when a deviation from the saddle-point entails a Pareto-improvement. This implies that implementability has to be checked ex-post with proposition 1 in this case.

In practice, the restriction that a tax schedule does not contain an internal maximum is rather weak. As can be seen from proposition 1, the first derivative of the tax system is defined by the wedge. As such, if none of the wedges change sign from positive to negative or vice versa the tax system is monotonic and it does not contain internal maxima. Most models in public finance satisfy this criteria because each good is usually either taxed or subsidized over the entire domain.

6.2.1 Mirrleesian Implementation

By proposition 3 a convex or monotonic tax system can implement an allocation provided it is optimal to a welfarist planner. The Mirrleesian planner is welfarist, such that a Mirrleesian separable tax system can implement this allocation provided wedges do not change sign from positive to negative and there is only one-dimensional heterogeneity. This is summarized in the next corollary

Corollary 1 *If $p = 1$ and the conditions of proposition 3 are met, the Mirrleesian tax system can implement the second-best.*

This result implies that a separable tax system can implement the second-best of a welfarist planner if there is only one source of heterogeneity. The restriction to single dimensional heterogeneity is due to the nature of the second-best wedges. If heterogeneity is multi-dimensional the wedges are non-separable (almost) by definition (e.g. Renes and Zoutman, 2013, Armstrong, 1996) such that the separability found in Mirrlees (1976, sect. 3) will be lost with multi-dimensional heterogeneity.

7 Concluding remarks

Our approach shows several results that are directly relevant to the literature on the optimal taxation problem and mechanism design in general. The proof of proposition 3 provides insights into implementation issues that are also relevant to other fields of mechanism design. A relatively broad class of tax systems implements the second-best of a welfarist planner because the objective of the planner is strictly increasing in agent's utility. In fields such as monopoly pricing and auction theory the objectives of the principal and the agents are opposed in the sense that an increase in a monopolist's profits (at fixed quantities) automatically comes at the expense of the consumers. As such, the implementation has to be much more restrictive in these fields (see also Armstrong (1996) and Renes (2011)).

This result highlights a unique feature of the Mirrleesian optimal tax model. Unlike the design problem of auctioneers and monopolists, the maximization of the mechanism designer is quite closely aligned with that of the agents he faces. This means the planner can let the agents maximize their utility with relatively little restrictions. This alignment also has interesting effects on the restrictions that are required for implementation. Restrictions will often contain interdependencies, increasing the complexity of the tax-schedule relatively quickly. These interdependencies, like wealth tests on income assistance in welfare states, are necessary to prevent rational people from taking advantage of subsidies that are not targeted at them.

Future work should focus on the possibility to extend this work to dynamically stochastic settings and on finding more tight descriptions of the ex-ante conditions.

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A Proofs

For bookkeeping, the Jacobian of first derivatives $\phi'(\cdot)$ of any function $\phi(\cdot) : \mathcal{R}^a \to \mathcal{R}^b$, is of dimension $b \times a$, while the second derivatives $\phi''(\cdot)$ are of dimension $ab \times a$. For any multi-vector functions $\psi(\mathbf{z}^1, \mathbf{z}^2, \ldots) : \mathcal{R}^{a^1} \times \mathcal{R}^{a^2} \ldots \to \mathcal{R}$ the vector of first derivatives ψ_{z^i} are of dimension $a^i \times 1$ and the matrix of second derivatives $\psi_{z^i z^j}$ are of dimension $a^i \times a^j$ where the dimension of the matrix follows the order of the subscripts. In addition, let superscript T be the transpose operator. Vectors and multi-dimensional constructs are denoted in bold-face, scalars are in normal-face.

A.1 Proof of proposition 1

Proof. Due to non-satiation of the utility function we know that the budget constraint will hold with equality such that we know that:

$$y^*(\mathbf{n}) = q(\mathbf{x}^*(\mathbf{n})) - T(\mathbf{x}^*(\mathbf{n}))$$

Direct substitution of the budget constraint into the utility function allows us to write the first-order conditions to problem (2) as:

$$\mathbf{0} = u_{\mathbf{x}} + (q' - T')^T u_y \tag{10}$$

which directly implies equations (7) and (8).

Now take the second-order derivative of the utility function with respect to $\mathbf{x}$ to get the second-order conditions:

$$u_{\mathbf{xx}} + \left(2u_{\mathbf{x}y} + u_{yy}\left(q'(\mathbf{x}^*) - T'(\mathbf{x}^*)\right)^T\right)\left(q'(\mathbf{x}^*) - T'(\mathbf{x}^*)\right) + u_y\left(q''(\mathbf{x}^*) - T''(\mathbf{x}^*)\right) \lessdot 0 \tag{11}$$

Differentiate the marginal rate of substitution, $\mathbf{s}$, to $\mathbf{x}$ using the definition of $\mathbf{s}$ and using the implicit function theorem to define $y(u, \mathbf{x}, \mathbf{n})$:

$$\frac{\partial \mathbf{s}(\mathbf{x}, y(u, \mathbf{x}, \mathbf{n}), \mathbf{n})}{\partial \mathbf{x}} = -\frac{\left(u_{\mathbf{xx}} + 2u_{\mathbf{x}y}\mathbf{s}^T\right) - u_{yy}\mathbf{s}\mathbf{s}^T}{u_y} \tag{12}$$

Now combining (8) with (12) allows us to simplify (11) and obtain the final codition:

$$\begin{aligned} -\left(\frac{\partial \mathbf{s}\left(\mathbf{x},y\left(u,\mathbf{x},\mathbf{n}\right),\mathbf{n}\right)}{\partial \mathbf{x}}+q''\left(\mathbf{x}^{*}\right)-T''\left(\mathbf{x}^{*}\right)\right)u_{y} &\nless 0 \Leftrightarrow \\ -\frac{\partial \mathbf{s}\left(\mathbf{x},y\left(u,\mathbf{x},\mathbf{n}\right),\mathbf{n}\right)}{\partial \mathbf{x}}+q''\left(\mathbf{x}^{*}\right)-T''\left(\mathbf{x}^{*}\right) &\nless 0 \end{aligned}$$

where the final step follows from the assumption that $u_y > 0$. ■

A.2 Proof to proposition 2

Equations (7) and (8) uniquely define the tax schedule for $\mathbf{x}^*(\mathbf{n})$ on its domain $\mathbf{N}$. If $\mathbf{x}^*(\mathbf{n})$ is bijective there is an unique inverse mapping $\mathbf{n}^*(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{X}$. Therefore, equations (7) and (8) define the tax schedule for $\mathbf{x}^*(\mathbf{n}^*(\mathbf{x}))$ on its domain $\mathbf{x} \in \mathbf{X}$. Hence, the tax schedule is defined on the entire choice space. Note that we do not need to check for second-order conditions (9) in this case, because we have assumed that the allocation $\mathbf{x}^*(\mathbf{n})$ is (second-order) incentive compatible for all $\mathbf{n} \in \mathbf{N}$. Therefore, the unique tax schedule that implements this allocation must also be implementable.

A.3 Proof to proposition 3

Suppose on the contrary that that (9) is not satisfied for some agent of type $\mathbf{n}$. Consider a deviation from the second-best allocation $\alpha\Delta\mathbf{x}$ where $\alpha > 0$ and $\Delta\mathbf{x}$ is a $k \times 1$ vector with length one. The utility gain of such a deviation can be approximated by a second-order Taylor expansion:

$$\begin{aligned} u\left(\mathbf{x}^{*}\left(\mathbf{n}\right)+\alpha\mathbf{\Delta x},q\left(\mathbf{x}^{*}\left(\mathbf{n}\right)+\alpha\Delta\mathbf{x}\right)-T\left(\mathbf{x}^{*}\left(\mathbf{n}\right)+\alpha\Delta\mathbf{x}\right),\mathbf{n}\right)-u^{*} &= \\ \left(u_{\mathbf{x}}^{T}+u_{y}\left(q'-T'\right)\right)\alpha\mathbf{\Delta x} &+ \\ \frac{1}{2}\alpha^{2}\mathbf{\Delta x}^{T}\left(u_{\mathbf{xx}}+\left(2u_{\mathbf{x}y}+u_{yy}\left(q'-T'\right)^{T}\right)\left(q'-T'\right)+u_{y}\left(q''-T''\right)\right)\mathbf{\Delta x} &= \\ \frac{1}{2}\alpha^{2}\mathbf{\Delta x}^{T}\left(u_{\mathbf{xx}}+\left(2u_{\mathbf{x}y}+u_{yy}\left(q'-T'\right)^{T}\right)\left(q'-T'\right)+u_{y}\left(q''-T''\right)\right)\mathbf{\Delta x} &= \\ \frac{1}{2}\alpha^{2}u_{y}\mathbf{\Delta x}^{T}\left(-\frac{\partial \mathbf{s}\left(\mathbf{x},y\left(u,\mathbf{x},\mathbf{n}\right),\mathbf{n}\right)}{\partial \mathbf{x}}+q''\left(\mathbf{x}^{*}\right)-T''\left(\mathbf{x}^{*}\right)\right)\mathbf{\Delta x} & \end{aligned}$$

where the first order terms equal zero by the first-order condition (8). Due to symmetry of the matrix of second order conditions for sufficiently small α the deviation strategy $\alpha\Delta\mathbf{x}$ and $-\alpha\Delta\mathbf{x}$ yield approximately the same utility. In addition, if $\left(-\frac{\partial \mathbf{s}(\mathbf{x},y(u,\mathbf{x},\mathbf{n}),\mathbf{n})}{\partial \mathbf{x}}+q''\left(\mathbf{x}^{*}\right)-T''\left(\mathbf{x}^{*}\right)\right)$ is not negative semi-definite there is at least one deviation strategy $\Delta\hat{\mathbf{x}}$ which yields a positive utility gain. The change in tax revenue due to such a deviation can also be found by means of a second-order Taylor expansion:

$$T\left(\mathbf{x}^*\left(\mathbf{n}\right)+\alpha\Delta\hat{\mathbf{x}}\right)-T\left(\mathbf{x}^*\left(\mathbf{n}\right)\right)\approx\alpha T'\Delta\hat{\mathbf{x}}+\frac{1}{2}\alpha^2\Delta\hat{\mathbf{x}}^T T''\Delta\hat{\mathbf{x}}.$$

The first-order term will always be non-negative for either strategy $-\Delta\hat{\mathbf{x}}$ or $\Delta\hat{\mathbf{x}}$. If for either choice it's positive, the first-order term dominates the second-order term for sufficiently small α and hence, the deviation results in higher tax revenue. If the first-order term is zero, we need to consider the second-order term. If it's negative apparently the tax schedule contains an internal maximum on the allocation in $\Delta\hat{\mathbf{x}}$ which violates our assumption. Therefore, if the first-order term is zero the second term must be non-negative. Hence, tax revenue always weakly increases in either $-\Delta\hat{\mathbf{x}}$ or $\Delta\hat{\mathbf{x}}$. Therefore, one of these deviations must be a Pareto-improvement and we run into a contradiction. If a Pareto-improvement over the allocation can be found within a particular implementation, then the original allocation could not have been second-best.

B Example

In figure 1 and 2, welfare function and resource constraint are equal to:

$$\begin{aligned}
u &= log(y)-\frac{1}{1.5}\left(\frac{x_1}{n}\right)^{1.5}-\frac{1}{1.5}\left(\frac{x_2}{n}\right)^{1.5} \\
W &= \int\left[u(n)+E(x_1.x_2)\right]dF(n) \\
E(x_1,x_2) &= \frac{1}{1.5}\left(\left(\frac{x_1(n)}{n}\right)^{1.5}+\left(\frac{x_2(n)}{n}\right)^{1.5}-\left(\frac{x_1(n)}{n}\right)^{1.5}*\left(\frac{x_2(n)}{n}\right)^{1.5}\right) \\
\int_N y(n)dF(n) &= \int_N x_1(n)+x_2(n)dF(n)
\end{aligned}$$

We assume that the type-space is uni-dimensional and the types are uniformly distributed over a closed interval on the real line. The first-order approach to this problem yields the allocation shown in the figures 1 2. This second-best allocation can only be implemented by the central planner if he uses interdependencies to map out the off-allocation consumption choices/coordinates. The planner can than determine what off-allocation points have to be taxed prohibitively to ensure that each individual prefers his own bundle over any other choice.